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HALL CHAINS IN NORMAL SUBGROUPS OF FINITE p-GROUPS

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ABSTRACT

Some sufficient conditions for existence of k-admissible Hall chains ($= \mathcal{H}_k$-chains) in normal subgroups of finite p-groups are established (for irregular p-groups we consider only the case $k = p$). In Propositions 13–15 we study p-groups without $\mathcal{H}_p$-chains, and metacyclic 2-groups with the above property are classified. Abelian p-groups with exactly one $\mathcal{H}_k$-chain are characterized in Proposition 12.

This note supplements [H, Theorem 2.5] and [B3, Theorem 1 and Supplement 2 to Theorem 1].

In what follows, G is a finite p-group, p is a prime, m, n, k, t are natural numbers and i is a nonnegative integer. We use notation and agreements standard for finite p-group theory, in particular, the bar convention (see [B2, B3]). We assume throughout this note that

($*$) $H > \{1\}$ is a normal subgroup of a p-group G.

We begin with the following definitions.

Definition 1: Given k, let

$$\mathcal{C} : \{1\} = L_0 < L_1 < \cdots < L_n = H$$

be a chain (of length $|\mathcal{C}| = n$) of G-invariant subgroups in H such that $\exp(L_i/L_{i-1}) = p$ and $|L_i/L_{i-1}| \leq p^k$ for $i = 1, \dots, n$. Then $\mathcal{C}$ is called a *k-admissible* chain in H. For this $\mathcal{C}$, set $i_0(\mathcal{C}) = \max\{i \geq 0 : |L_i| = p^{ki}\}$.

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It follows that $|L_{i_0(\mathcal{C})+j}| < p^{k(i_0(\mathcal{C})+j)}$ for all $j > 0$. In general, the sequence $\{|L_1|, |L_2 : L_1|.|L_3 : L_2|, \dots\}$ is not monotone as, for $k = 2$, the metacyclic group $G = \langle a, b \mid a^8 = 1, a^4 = b^4, a^b = a^{-1}\rangle$ shows (for this G, the sequence of indices of $\mathcal{C}$ is $4, 2, 4$). We have $\exp(L_i) \le p^i$ for all $i \le n$. In the sequel, $\mathcal{C}$ is as in Definition 1.

Definition 2: A k-admissible chain $\mathcal{C}$ in H *dominates* over a k-admissible chain

$$\mathcal{C}_1 : \{1\} = M_0 < M_1 < \cdots < M_s = H$$

if, with respect to lexicographic ordering, the sequence

$$\{|L_1|, |L_2 : L_1|, \dots, |L_n : L_{n-1}|\}$$

is greater or equal than the sequence $\{|M_1|, |M_1 : M_0|, \dots, |M_s : M_{s-1}|\}$. In that case, we write $\mathcal{C} \ge \mathcal{C}_1$. If, in addition, $|L_t : L_{t-1}| > |M_t : M_{t-1}|$ for some t, but $|L_i : L_{i-1}| = |M_i : M_{i-1}|$ for all $i < t$, we write $|\mathcal{C}| > |\mathcal{C}_1|$ (in that case, $\mathcal{C}$ dominates strongly over $\mathcal{C}_1$).

In the sequel, $\mathcal{C}_1$ is such as in Definition 2.

Definition 3: A k-admissible chain $\mathcal{C}$ in H is said to be *dominating* if, for all k-admissible chains $\mathcal{C}_1$ in H, we have $\mathcal{C} \ge \mathcal{C}_1$.

Thus, two k-admissible dominating chains in H have the same sequence of indices. We also consider k-admissible chains in G-invariant subgroups $A < H$ and in H/A. There is a k-admissible dominating chain in H always.

Definition 4: A k-admissible chain $\mathcal{C}$ in H is said to be a *Hall chain* (or $\mathcal{H}_k$-chain, for brevity), if, for all $j > 0$, we have $L_{i_0+j} = \Omega_{i_0+j}(H)$, where $i_0 = i_0(\mathcal{C})$.

Let $p = 2$ and let H be dihedral of order 8; then H has an $\mathcal{H}_2$-chain. Now let $H < G$, where G is dihedral of order 16. Then there are no $\mathcal{H}_2$-chains in H (as a normal subgroup of G).

If a p-group G has an $\mathcal{H}_k$-chain $\mathcal{C}$ and $\Omega_1(G) = G$, then all indices of the chain $\mathcal{C}$ except for the last one are equal to p^k. Indeed, if $|L_i| < p^{ki}$, then $L = \Omega_i(G) = G$ so $i = n$. If H has a G-invariant subgroup R of order p^k and exponent p such that $\exp(H/R) = p$, then H has an $\mathcal{H}_k$-chain.

All p-groups of order p^{2p} and exponent p^2 have $\mathcal{H}_k$-chains (notice that p-groups of maximal class and order p^{2p}, which have no $\mathcal{H}_p$-chains, are of exponent p^3). To prove this, consider the quotient group $\bar{G} = G/\mho_1(G)$. Since absolutely regular p-groups of order p^{2p} have exponent $> p^2$, we get $|\bar{G}| \geq p^p$ so $|\mho_1(G)| \leq p^p$. It follows that $\mho_1(G)$ is regular. By hypothesis, $\mho_1(G)$ is generated by elements of order p so $\exp(\mho_1(G)) \leq p$. Then, by Lemma 6, below, there is $R \triangleleft G$ of order p^p and exponent p such that $\mho_1(G) \leq R$. Then $\{1\} < R < G$ is the $\mathcal{H}_p$-chain in G.

If, in Definition 4, $L_{i_0+1} < H$, then $\exp(L_{i_0+1}) = p^{i_0+1}$, and so $\exp(L_i) = p^i$ for all $i \leq n$. If $i_0(\mathcal{C}) \geq n-1$, then a k-admissible dominating chain $\mathcal{C}$ must be an $\mathcal{H}_k$-chain. As Lemma 3(a) shows, $\mathcal{H}_k$-chains are k-admissible dominating, however, the converse is not true in general. It is asserted in [H, Theorem 2.5] and [B3, Supplement 1 to Theorem 1] that there exists in H an $\mathcal{H}_k$-chain for $k \leq p-1$, however, this is not true for $k = p$ (see Remark 3, below). Some conditions guaranteing existence of $\mathcal{H}_p$-chains in H are stated in Theorem 10. Theorem 11 shows, in particular, that there is in regular H an $\mathcal{H}_k$-chain for any k. Proposition 12 asserts that an abelian p-group G has only one $\mathcal{H}_k$-chain if and only if $|\Omega_1(G)| \leq p^k$. In Propositions 13 and 14 we study the p-groups without $\mathcal{H}_p$-chains. The set $\mathcal{M}_0$, introduced there, is an important invariant of G.

If $\mathcal{C}$ and $\mathcal{C}_1$ are $\mathcal{H}_k$-chains in H, then $L_{i_0+u} = \Omega_{i_0+u}(H) = M_{i_0+u}$, where $i_0 = i_0(\mathcal{C})$ and $u \geq 1$.

Example: Each 2-group of order $\leq 2^3$ has an $\mathcal{H}_2$-chain. If G be a 2-group of maximal class and order $\geq 2^4$ then, by Remark 3, below, G has no $\mathcal{H}_2$-chain. (i) We claim that if a group G of order 2^4 is not of maximal class, it has an $\mathcal{H}_2$-chain. Indeed, let $R \triangleleft G$ be abelian of type $(2,2)$ (R exists, by Lemma 7). If G/R is noncyclic, then $\{1\} < R < G$ is the desired $\mathcal{H}_2$-chain. Now let G/R be cyclic. If G has a cyclic subgroup of index 2, then $R = \Omega_1(G)$ and $\{1\} < R < \Omega_2(G) < G$ is the desired $\mathcal{H}_2$-chain. If G has no cyclic subgroup of index 2, then $G = C \cdot R$, where C is cyclic of order 4. Let $U \leq R \cap \mathrm{Z}(G)$ be of order 2 and $R_1 = U \times \Omega_1(C)(\leq \mathrm{Z}(G)_)$; then G/R_1 is abelian of type $(2,2)$ and so $\{1\} < R_1 < G$ is the desired $\mathcal{H}_2$-chain. (ii) A 2-group G of order $> 2^4$ with cyclic subgroup of index 2, which is not of maximal class, has the unique $\mathcal{H}_2$-chain $\{1\} < \Omega_1(G) < \Omega_2(G) < \cdots < G$ as follows from classification of such groups. (iii) We claim that a 2-group G of order 2^5, which is not of maximal class, has an $\mathcal{H}_2$-chain. In view of (ii), we may assume that G has no cyclic subgroup of index 2. Let $R \triangleleft G$ be abelian of type $(2,2)$. If $H/R < G/R$ is abelian of type $(2,2)$, then $\{1\} < R < H < G$ is an $\mathcal{H}_2$-chain.

Now assume that G/R has no abelian subgroup of type $(2,2)$. Then G/R is either cyclic or $\cong \mathrm{Q}_8$, the ordinary quaternion group. (1iii) Let G/R be cyclic. Then $G = Z \cdot R$ is a semidirect product, where Z is cyclic of order 8. In that case, the subgroup $R_0 = R_1 \times \Omega_1(Z) \lhd G$, where $R_1 \le R \cap \mathrm{Z}(G)$ is of order 2, is abelian of type $(2,2)$ and G/R_0 has an abelian subgroup of type $(2,2)$; then G has an $\mathcal{H}_2$-chain, as above. (2iii) Now let $G/R \cong \mathrm{Q}_8$. If $\Omega_1(G) = R$, then $\{1\} < \Omega_1(G) < \Omega_2(G) < G$ is the unique $\mathcal{H}_2$-chain in G. It remains to consider the case where $\Omega_1(G) = U$ is elementary abelian of order 8; then $\exp(G) = 4$. Let $F/R < G/R$ be of order 4; then $F = L \cdot R$, where L is cyclic of order 4 and $R \cap L = \{1\}$. Let $K \le R \cap \mathrm{Z}(G)$ be of order 2. Set $R_1 = K \times \Omega_1(L) (\le \mathrm{Z}(F))$; then $R_1 \lhd G$ and $\{1\} < R_1 < F < G$ is the desired $\mathcal{H}_2$-chain in G.

It is easy to see that every minimal nonabelian p-group has an $\mathcal{H}_k$-chain for all k.

We are interested in the following statements concerning a p-group G and all n:

1. Each element of $\mho_n(G)$ is a p^n-th power, i.e., $\mho_n(G) = \{x^{p^n} : x \in G\}$.
2. $\exp(\Omega_n(G)) \le p^n$, i.e., $\Omega_n(G) = \{x \in G \colon o(x) \le p^n\}$.
3. $|\Omega_n(G)| = |G : \mho_n(G)|$.

Definition 5: For $i \in \{1,2,3\}$, a p-group G is called a $\mathcal{P}_i$-group if all sections of G satisfy condition (i) for all n, and G is called a $\mathcal{P}$-group, if it is a $\mathcal{P}_i$-group for $i = 1,2,3$ simultaneously.

In his important paper [M], Avinoam Mann has studied the interrelations between the above defined properties $\mathcal{P}_i$, $i = 1,2,3$, and $\mathcal{P}$ in detail. The following unexpected result holds [M]: $\mathcal{P}_3 \subset \mathcal{P}_2 \subset \mathcal{P}_1$ so that $\mathcal{P}_3$-groups coincide with $\mathcal{P}$-groups (however, in what follows, we do not use this deep result). Regular p-groups are $\mathcal{P}$-groups (Philip Hall). A Sylow 2-subgroup of the Suzuki simple group $\mathrm{Sz}(2^{2n+1})$ satisfies conditions 1–3 but it is not a $\mathcal{P}_3$-group since it has a nonabelian section of order 8, which has no property (3). For $p = 2$, the irregular group $\mathrm{M}_{2^{n+1}} = \langle x, y \mid x^{2^n} = y^2 = 1, x^y = x^{1+2^{n-1}}, n > 2 \rangle$ is a $\mathcal{P}$-group. For $p > 2$, Mann [M] constructed an irregular group X of order p^{p+1} such that $|\Omega_1(X)| = p^p$ (this X is a $\mathcal{P}$-group). All p-groups of maximal class are $\mathcal{P}_1$-groups as follows from [B5]. A p-group of maximal class and order $> p^{p+1}$ is not a $\mathcal{P}_2$-group (see Lemma 5(b), below).

See [Hup, Kap III] on p-groups with cyclic subgroup of index p, regular p-groups and p-groups of maximal class

Remark 1: For a $\mathcal{P}$-group G, the following assertions hold:

(i) $$|\Omega_1(G)| \geq |\Omega_2(G)/\Omega_1(G)| \geq |\Omega_3(G)/\Omega_2(G)| \geq \cdots,$$

(ii) $$|G/\mho_1(G)| \geq |\mho_1(G)/\mho_2(G)| \geq |\mho_2(G)/\mho_3(G)| \geq \cdots.$$

We have $\exp(\Omega_1(G)) = \exp(\Omega_2(G)/\Omega_1(G)) = p$ so $\mho_1(\Omega_2(G)) \leq \Omega_1(G)$ and

$$|\Omega_1(G)| = |\Omega_1(\Omega_2(G))| = |\Omega_2(G)/\mho_1(\Omega_2(G))| \geq |\Omega_2(G)/\Omega_1(G)|.$$

We have, for $k > 1$, $\Omega_k(G)/\Omega_{k-1}(G) = \Omega_1(G/\Omega_{k-1}(G))$. Therefore, (i) follows by induction on k. As to (ii), we have

$$|G/\mho_1(G)| = |\Omega_1(G)| \geq |\Omega_2(G)/\Omega_1(G)| = \frac{|G : \mho_2(G)|}{|G : \mho_1(G)|} = |\mho_1(G)/\mho_2(G)|.$$

We suggest to the reader to finish the proof of (ii). The groups satisfying (i) and (ii), are called upper and lower pyramidal, respectively. Next we prove that if $A < G$, then $|A/\mho_1(A)| = |\Omega_1(A)| \leq |\Omega_1(G)| = |G/\mho_1(G)|$ since G is a $\mathcal{P}_3$-group.

We use the following fact freely: if $\exp(G) = p^e$, then $\exp(G/\Omega_1(G)) \leq p^{e-1}$ since $\mho_{e-1}(G) \leq \Omega_1(G)$.

A p-group G is said to be **absolutely regular** (Blackburn) if $|G/\mho_1(G)| < p^p$. By Hall's regularity criterion [H, Theorem 2.3], absolutely regular p-groups are regular. All sections of absolutely regular p-groups are absolutely regular (Remark 1(iii)). All p-groups of class $< p$ (so groups of order $\leq p^p$) are regular. According Mann's letter, G is regular provided $G/\mathrm{Z}(G)$ is absolutely regular (for the proof, see [B1, Remark 7.2]).

For main properties of p-groups of maximal class, which we use in what follows, see [B5].

Lemma 1: *Let $F > \{1\}$ be a normal subgroup of a p-group G and let K be a G-invariant subgroup of order p in F. Write $\bar{G} = G/K$.*

(a) *Suppose that $\{\bar{1}\} = \bar{F}_0 < \bar{F}_1 < \cdots < \bar{F}_n = \bar{F}$ is an $\mathcal{H}_p$-chain in $\bar{F}$ such that $|\bar{F}_i| = p^{pi}$ for $i = 1, \ldots, n$. If every section of F of order p^{p+1} has a characteristic subgroup of order p^p and exponent p, then there exists in F an $\mathcal{H}_p$-chain $\{1\} = L_0 < L_1 < \cdots < L_n < L_{n+1} = F$ such that $|F_i : L_i| = p$ for $i = 0, 1, \cdots, n$ so that L_n is of order p^{pn} and exponent $\leq p^n$.*

(b) *Let k be fixed and suppose that all sections of F of order p^{k+1} are $\mathcal{P}_3$-groups. Suppose that $\{\bar{1}\} = \bar{F}_0 < \bar{F}_1 < \cdots < \bar{F}_n = \bar{F}$ is an $\mathcal{H}_k$-chain in $\bar{F}$ such that $|\bar{F}_i| = p^{ki}$ for $i = 1, \ldots, n$. Then there exists in F an $\mathcal{H}_k$-chain $\{1\} = L_0 < L_1 < \cdots < L_n < L_{n+1} = F$ such that $|F_i : L_i| = p$ for $i = 0, 1, \ldots, n$ so that L_n is of order p^{kn} and exponent $\le p^n$.*

Proof. We proceed by induction on $|F|$.

(a) By hypothesis, F_1 is of order p^{p+1} and F_1/K is of order p^p and exponent p. Suppose that F_1 is irregular. Then F_1 has a characteristic subgroup of order p^p and exponent p, and we denote that subgroup by L_1. Now suppose that F_1 is regular. Then $\mho_1(F_1) \le K$ so $|\Omega_1(F_1)| = |F_1/\mho_1(F_1)| \ge |F_1/K| = p^p$, and we conclude that $\Omega_1(F_1)$ is of order $\ge p^p$ and exponent p. In this case, we take L_1 to be an arbitrary G-invariant subgroup of order p^p in $\Omega_1(F_1)$. If $F_1 = F$, we are done, so we let $F_1 < F$. The group F/L_1 is an extension of F_1/L_1 of order p by F/F_1 of order $p^{p(n-1)}$. By induction, there is an $\mathcal{H}_p$-chain $L_1/L_1 < L_2/L_1 < \cdots < L_n/L_1 < F/L_1$ such that $|(F_i/L_1) : (L_i/L_1)| = p$ for all $i = 2, \cdots, n$. Then $\{1\} = L_0 < L_1 < \cdots < L_n < F$ is the desired $\mathcal{H}_p$-chain.[1] (b) is proved in the same way as (a). ∎

Lemma 2: *Let $F = \Omega_n(H)$ and let $\mathcal{C} : \{1\} = L_0 < L_1 < \cdots < L_n = F$ be an $\mathcal{H}_k$-chain of length n in F.*

(a) *Suppose that $\mathcal{C}_1 : \{1\} < L_{n+1}/L_n < L_{n+2}/L_n < \cdots < L_{n+m}/L_n = H/L_n$, where $L_{n+i}/L_n = \Omega_i(H/L_n)$, is an $\mathcal{H}_k$-chain in the quotient group $H/L_n = H/F$. Then $\mathcal{C}_2 : \{1\} = L_0 < L_1 < L_n = F < L_{n+1} < \cdots < L_{n+m} = H$ is an $\mathcal{H}_k$-chain in H.*

(b) *If, in addition, $k = p$ and H/F is absolutely regular, then the chain $\mathcal{C}_2$ from* (a) is an $\mathcal{H}_p$-chain.

Proof. (b) follows from (a) immediately since H/F has an $\mathcal{H}_p$-chain with $L_{n+i}/L_n = \Omega_i(H/L_n)$. It remains to prove (a).

One may assume that $F < H$; then $\exp(H) > p^n$ so, since $\exp(F)$ cannot be $> p^n$, we get $\exp(F) = p^n$. For $j \le n$, we have $\Omega_j(F) = \Omega_j(\Omega_n(H)) = \Omega_j(H)$. To prove that $\mathcal{C}_2$ is an $\mathcal{H}_k$-chain, it suffices to show that $L_{n+i} = \Omega_{n+i}(H)$ for $i \le m$. Take $x \in H$ with $o(x) \le p^{n+i}$. We have to prove that $x \in L_{n+i}$. It follows from $F = L_n = \Omega_n(H)$ and $\exp(L_n) = p^n$ that $\langle x\rangle \cap L_n = \Omega_n(\langle x\rangle)$ so (in H/L_n) we have $o(xL_n) \le p^i$, and hence $xL_n \in \Omega_i(H/L_n) = L_{n+i}/L_n$. ∎

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[1] The same conclusion holds if every irregular section of F of order p^{p+1} is an $\mathcal{P}_3$-group.

Lemma 3: *Suppose that there is an $\mathcal{H}_k$-chain $\mathcal{C}$ (as in Definition 1) in H. Then*

(a) *$\mathcal{C}$ is a k-admissible dominating chain,*
(b) *all k-admissible dominating chains in H are $\mathcal{H}_k$-chains.*

Proof. Suppose that all considered chains are k-admissible.

(a) Let $\mathcal{C}_1$ (as in Definition 2) be a dominating chain in H. We have to prove that $|L_i| = |M_i|$ for all i. Let t be such that $|M_t| < p^{kt}$. We have $|L_t| \le |M_t| < p^{kt}$ so $L_t = \Omega_t(H)$ since $\mathcal{C}$ is an $\mathcal{H}_k$-chain. Since $\exp(M_t) \le p^t$, we get $M_t \le \Omega_t(H) = L_t$ so $L_t = M_t$. Next assume that $|L_s| < p^{ks}$; then $L_s = \Omega_s(H) \ge M_s$ so, since the chain $\mathcal{C}_1$ is dominating, we get $M_s = L_s$. Now let $|M_u| = p^{pu}$. Then, by the above, $|L_u| = p^{pu}$, completing the proof of (a).

(b) Let $\mathcal{C}_1$ be a dominating chain in H; we have to show that $\mathcal{C}_1$ is also an $\mathcal{H}_k$-chain. By (a), the chain $\mathcal{C}$ is dominating so $|L_i| = |M_i|$ for all i, and we get $i_0 = i_0(\mathcal{C}) = i_0(\mathcal{C}_1)$. Since $\mathcal{C}$ is an $\mathcal{H}_k$-chain and $\exp(M_{i_0+u}) \le p^{i_0+u}$, we get $L_{i_0+u} = \Omega_{i_0+u}(H) \ge M_{i_0+u}$ for all $u > 0$ so $M_{i_0+u} = \Omega_{i_0+u}(H) = L_{i_0+u|}$ since $\mathcal{C}_1$ is dominating, and we are done. ∎

The point of Lemma 3(b) is that if we want to prove that all k-admissible dominating chains in H are $\mathcal{H}_k$-chains, it suffices to show that at least one of these chains is an $\mathcal{H}_k$-chain.

Lemma 4: *Let $\mathcal{C}$ be a k-admissible dominating chain in H. Set $\bar{G} = G/L_1$. Then $\overline{\mathcal{C}} : \{\bar{1}\} < \bar{L}_2 < \cdots < \bar{L}_n = \bar{H}$ is a k-admissible dominating chain in $\bar{H}$.*

Proof. Indeed, suppose that $\overline{\mathcal{C}}_1$ is a k-admissible dominating chain in $\bar{H}$ and assume that $\overline{\mathcal{C}}_1 > \overline{\mathcal{C}}$; then the chain $\mathcal{C}_1$, which is the 'inverse image' of the chain $\overline{\mathcal{C}}_1$, is k-admissible and satisfies $\mathcal{C}_1 > \mathcal{C}$, a contradiction. ∎

If $\overline{\mathcal{C}}$ is a k-admissible dominating chain in $\bar{G} = G/N$, where N is of order p^k and exponent p, then its inverse image $\mathcal{C}$ is k-admissible but can be not dominating in G. Indeed, let $G = U \times V \times W$, where U, V and W are cyclic of orders p, p, p^2, respectively. Let $k = 2$ and $L_1 = U \times V$, $\bar{G} = G/L_1$. Then the chain $\mathcal{C}$ with $L_i = L_1 \times \Omega_{i-1}(W)$, $i = 1, 2, 3$, is not an $\mathcal{H}_2$-chain in G although $\{\bar{1}\} < \bar{L}_2 < \bar{L}_3 = \bar{G}$ is an $\mathcal{H}_2$-chain in the cyclic p-group $\bar{G}$. Indeed, if $M_1 = U \times \Omega_1(W)$ and $M_2 = G$, then the chain $\mathcal{C}_1 : \{1\} < M_1 < M_2 = G$ dominates over $\mathcal{C}$.

LEMMA 5: *Let G be an irregular p-group of maximal class. Then all sections of G of order p^{p+1} are $\mathcal{P}_3$-groups ($\mathcal{P}_2$-groups) if and only if $|G| = p^{p+1}$ and $|\Omega_1(G)| = p^p$ ($\Omega_1(G) < G$).*

Lemma 5 follows easily from Blackburn's theory of p-groups of maximal class (see [B5]).

Proof. Assume that $|G| > p^{p+1}$; then $G/\mathrm{K}_{p+1}(G)$, where $\mathrm{K}_{p+1}(G)$ is the $(p+1)$-st member of the lower central series of G, is neither $\mathcal{P}_3$- no $\mathcal{P}_2$-group. It remains consider the case $|G| = p^{p+1}$. If G is a $\mathcal{P}_3$-group, then $|\Omega_1(G)| = |G/\mho_1(G)| = p^p$. Next, G is a $\mathcal{P}_2$-group if and only if $\Omega_1(G) < G$. ∎

LEMMA 6: *If H has a G-invariant subgroup B of order p^p and exponent p, then each maximal G-invariant subgroup of H of exponent p is of order $\geq p^p$.*

Proof. Let A be a maximal G-invariant subgroup of exponent p in H. We have to prove that $|A| \geq p^p$. Assume that this is false; then $A \not\leq B$. Let $R \leq B$ be the least G-invariant subgroup such that $R \not\leq A$. Then $|A| < |AR| = p|A| \leq p^p$ so AR is regular. It follows from $\Omega_1(AR) = AR$ that $\exp(AR) = p$, contrary to the choice of A. (Similarly, under hypothesis of Lemma 6, every maximal subgroup of exponent p in H has order $\geq p^p$.) ∎

LEMMA 7 (B2, §7, Remark 2): *Suppose that the p-group G is neither absolutely regular nor of maximal class. Then the number of subgroups of order p^p and exponent p in G is $\equiv 1 \pmod p$ (so G has a normal subgroup of order p^p and exponent p).*

Let G be a p-group. Set $\mho^1(G) = \mho_1(G)$. If $\mho^i(G)$ has been defined, we set $\mho^{i+1}(G) = \mho_1(\mho^i(G))$. Since $\exp(G/\mho^k(G)) \leq p^k$, we get $\mho_k(G) \leq \mho^k(G)$ for all k. If G is a $\mathcal{P}$-group of exponent p^e, then $\mho^i(G) = \mho_i(G)$ for all i.

LEMMA 8: *Let $|G| = p^m$, $\exp(G) \leq p^e$ and $m \leq ke$.*

(a) *If G is lower pyramidal, then $\mho^{e-1}(G)$ is of order $\leq p^k$, and if $m < ke$, then $|\mho^{e-1}(G)| < p^k$. If, in addition, G is a $\mathcal{P}_2$-group, then $\exp(\mho_{e-1}(G)) \leq p$.*

(b) *If $k = p$, then $\mho^{e-1}(G)$ is either absolutely regular or of order p^p and exponent $\leq p$. In either case, $\mho_{e-1}(G)(\leq \mho^{e-1}(G))$ is of exponent p. If, in addition, $m < pe$, then we have $|\Omega_1(\mho^{e-1}(G))| < p^p$ so $\mho_{e-1}(G)$ is of order $\leq p^{p-1}$ and exponent p.*

Proof. (a) is obvious (the last assertion is true since $\mho_{e-1}(G) \le \Omega_1(G)$). For the proof of (b), see [B4, Lemmas 4, 6]. ∎

The following two assertions hold. (a) Let G be an absolutely regular p-group. If $|G| > p^{(p-1)k}$, then $\exp(G) > p^k$. (b) Now let G be a p-group of maximal class and order p^m, $m > p+1$. If $m-1 = (p-1)k$, then $\exp(G) = p^k$. If $m-1 > (p-1)k$, then $\exp(G) > p^k$. Assertion (a) follows since absolutely regular p-groups are pyramidal (Remark 1). As to (b), a p-group G of maximal class has an absolutely regular subgroup G_1 of order p^{m-1}, and $\exp(G_1) = \exp(G)$ [B5]; so the result follows from (a).

Lemma 9: (a) Let H be of order p^{pe} and exponent $\le p^e$. Then all indices of any p-admissible dominating chain in H equal p^p.

(b) Let H be a pyramidal (see Remark 1) $\mathcal{P}_2$-subgroup of order p^{ke} and exponent $\le p^e$. Then all indices of any k-admissible dominating chain in H equal p^k.

Proof. One may assume that $e > 1$. We use induction on $|H|$.

(a) By the paragraph preceding the lemma, H is neither absolutely regular nor of maximal class. Therefore, by Lemma 7, H has a G-invariant subgroup, say R, of order p^p and exponent p. Suppose that $\exp(H) \le p^{e-1}$. Then $\exp(H/R) \le \exp(H) \le p^{e-1}$ and $|H/R| = p^{p(e-1)}$. Therefore, by induction, there is in H/R an $\mathcal{H}_p$-chain $\{1\} = R/R = T_1/R < T_2/R < \cdots < T_e/R = H/R$, and all indices of this chain are equal to p^p. In that case, $\{1\} < T_1 < \cdots < T_e = H$ is the desired $\mathcal{H}_p$-chain. Therefore, one may assume, in what follows, that $\exp(H) = p^e$. In that case, $\mho_{e-1}(H)$ is of order $\le p^p$ and exponent $\le p$ (Lemma 8(b)) so $\mho_{e-1}(H) \le R$, where $R < H$ is a G-invariant subgroup of order p^p and exponent p (Lemma 6). Then H/R is a normal subgroup of order $p^{p(e-1)}$ and exponent $\le p^{e-1}$ in G/R. By induction, there exists an $\mathcal{H}_p$-chain $\{1\} = R/R = L_1/R < L_2/R < \cdots < L_e/R = H/R$ in H/R such that all indices of this chain equal p^p, and so $\{1\} < R = L_1 < L_2 < \cdots < L_e = H$ is the desired $\mathcal{H}_p$-chain.

(b) As in (a), one can assume that $\exp(H) = p^e$. Since H is lower pyramidal, $|\mho_{e-1}(H)| \le p^k$. Since $\mho_{e-1}(H)$ is generated by elements of order $\le p$ and H is a $\mathcal{P}_2$-group, we get $\exp(\mho_{e-1}(H)) \le \exp(\Omega_1(H)) = p$. Since H is upper pyramidal, we get $|\Omega_1(H)| \geq p^k$. Let $\mho_{e-1}(H) \leq R \leq \Omega_1(H)$, where R is G-invariant of order p^k. Then H/R is of order $p^{k(e-1)}$ and exponent $\leq p^{e-1}$. Now the result follows by induction in H/R, as in (a). ∎

Remark 2: Let H be a normal subgroup of order $\leq p^{ke}$ and exponent p^e in a p-group G. Suppose that there exists an $\mathcal{H}_k$-chain $\mathcal{C}$ in H (as in Definition 1). We claim that then $n = e$. Indeed, this is trivial for $e = 1$ so we let $e > 1$. Clearly, $n \geq e$. Assume that $n > e$. Then $|L_{n-1}| < |H| \leq p^{ke} \leq p^{k(n-1)}$ so $\Omega_{n-1}(H) = L_{n-1} < H$ since $\mathcal{C}$ is an $\mathcal{H}_k$-chain. It follows that $\exp(L_{n-1}) = p^{n-1} \geq p^e$, a contradiction, since $\Omega_{n-1}(H) = H$.

Let G be a group of order p^m possessing an $\mathcal{H}_k$-chain, say $\mathcal{C} : \{1\} = L_0 < L_1 < \cdots < L_{i_0} < \cdots < L_n = G$. Let $m = [m/k]k + s$, where $s < k$. Assume that $|\Omega_t(G)| \geq p^{kt}$ for all $t \leq [m/k]$. We claim that then $i_0 = i_0(\mathcal{C}) = [m/k]$, where $[x]$ is the integer part of the real number x. Clearly, $i_0 \leq [m/k]$. Assume that $i_0 < [m/k]$. Then $|\Omega_{i_0+1}(G)| < p^{k(i_0+1)}$ so, since $i_0 + 1 \leq [m/k]$, we get $|\Omega_{[m/k]}(G)| < p^{[m/k]k}$, contrary to the assumption. Otherwise, $i_0(\mathcal{C})$ equals such t that $|\Omega_t(G)| \geq p^{kt}$ and $|\Omega_{t+1}(G)| < p^{k(t+1)}$.

Let $\mathcal{C}$ be an $\mathcal{H}_k$-chain of length n in H with $i_0 = i_0(\mathcal{C})$. Suppose that $\exp(L_{i_0}) < p^{i_0}$. We claim that then $n \leq i_0 + 1$. Assume that $n > i_0 + 1$. We have $\exp(L_{i_0+1}) < p^{i_0+1}$. Since $\Omega_{i_0+1}(H) = L_{i_0+1} < H$, it follows that $\exp(L_{i_0+1}) = p^{i_0+1}$, contrary to what has just been said.

Remark 3: Let G be an irregular p-group of maximal class. (i) If G has an $\mathcal{H}_p$-chain, then either $|G| = p^{p+1}$ or $p^{p+1} < |G| < p^{2p}$ and $|\Omega_1(G)| = p^{p-1}$. (ii) Conversely, if $|\Omega_1(G)| = p^{p-1}$ and $|G| < p^{2p}$, then G has the unique $\mathcal{H}_p$-chain $\{1\} < \Omega_1(G) < G$ of length 2. Let us prove these assertions using [B5]. Let $\mathcal{C}$ be an $\mathcal{H}_p$-chain in G and assume that $|G| > p^{p+1}$. Since G has no normal subgroup of order p^p and exponent p, we get $|\Omega_1(G)| = p^{p-1}$. Since $|L_2| < p^{2p}$, we get $L_2 = \Omega_2(G)$. Since $\Omega_2(G) = G$ [B2, Remark 7.8], we have $|G| < p^{2p}$. In that case, $\exp(G/L_1) = p$ is of order $\leq p^p$ and exponent p so $\mathcal{C} : \{1\} < \Omega_1(G) < G$. Next, any group G of order p^{p+1} has an $\mathcal{H}_p$-chain. This is obvious if G is regular [Hup, §III.14]. If G is irregular, $\Phi(G)$ is of order p^{p-1} and exponent p. If $\Omega_1(G) = \Phi(G)$, then $\{1\} < \Phi(G) < G$ is an $\mathcal{H}_p$-chain. If $\Omega_1(G) > \Phi(G)$, then G has a maximal subgroup M of exponent p; then $\{1\} < M < G$ is an $\mathcal{H}_p$-chain. Now assertion (ii) is obvious. (Any regular p-group of maximal class has an $\mathcal{H}_k$-chain; see [Hup, §III.14] or Theorem 11, below.)

THEOREM 10: *Let $H > \{1\}$ be a normal $\mathcal{P}_2$-subgroup of a p-group G. Suppose, in addition, that every irregular section of H of order p^{p+1} has a characteristic subgroup of order p^p and exponent p. Then there exists in H a chain $\mathcal{C} : \{1\} = L_0 < L_1 < \cdots < L_n = H$ of G-invariant subgroups with the following properties ($i = 1, \ldots, n$):*

(a) *L_i/L_{i-1} is of order $\leq p^p$ and exponent p, and*
(b) *either $|L_i| = p^{pi}$ or else $L_i = \Omega_i(H)$.*

In other words there is in H an $\mathcal{H}_p$-chain. Then it follows from Lemma 3(b) that each p-admissible dominating chain in H is an $\mathcal{H}_p$-chain. If, in our theorem, $p = 2$, then the subgroup H is powerful, i.e., $H/\mho_2(H)$ is abelian [LM]. To prove this, we suppose that H is a minimal counterexample. By hypothesis, all sections of H of order 8 are abelian so H is also modular (Iwasawa). One may assume that $\exp(H) = 4$. Then H is minimal nonabelian so $|H| \leq 2^5$. It follows from Redei's classification of minimal nonabelian 2-groups that H has a nonabelian epimorphic image of order 8, a contradiction. It follows from properties of powerful p-groups that H is pyramidal. In what follows, however, we do not use the above results. Also notice that if U is an irregular section of G of order p^{p+1}, then $|\Omega_1(U)| = |U/\mho_1(U)| = p^p$.

Proof. If $\{1\} < N \leq H$ is G-invariant, then the pairs $H/N \leq G/N$ and $N \leq G$ satisfy the hypothesis. We proceed by induction on $|H|$. Let $\exp(H) = p^e$. One may assume that $e > 1$. Obviously, all G-invariant sections of H satisfy the hypothesis.

(i) Suppose that H has no G-invariant subgroup of order p^p and exponent p. Then H is either absolutely regular so $\{1\} = \Omega_0(H) < \Omega_1(H) < \cdots < \Omega_e(H) = H$ is an $\mathcal{H}_p$-chain in H, or irregular of maximal class and order p^{p+1} (Lemmas 7 and 5); in the last case, H must have a characteristic subgroup M of order p^p and exponent p, contrary to the assumption.

In what follows we assume that H has a G-invariant subgroup of order p^p and exponent p. Let $F_0 < H$ be a G-invariant subgroup of order p and set $\bar{G} = G/F_0$. By induction, there exists an $\mathcal{H}_p$-chain

$$\overline{\mathcal{C}}' : \{\bar{1}\} = \bar{F}_0 < \bar{F}_1 < \cdots < \bar{F}_n = \bar{H}$$

in $\bar{H}$. Write $i_0 = i_0(\overline{\mathcal{C}}')$. By Lemma 1, there exists an $\mathcal{H}_p$-chain

$$\mathcal{C}'' : \{1\} = L_0 < L_1 < \cdots < L_{i_0} < F_{i_0}$$

in F_{i_0} with $|F_{i_0} : L_{i_0}| = p$, so that

(α) F_{i_0} contains a G-invariant subgroup L_{i_0} of order p^{pi_0} and exponent $\leq p^{i_0}$, $i_0 > 0$.

One may assume that $i_0 < n$ (otherwise, $\mathcal{C}''$ is the desired chain) so $F_{i_0} < F_{i_0+1}$. Next, $H/F_{i_0}(\cong \bar{H}/\bar{F}_{i_0})$ has no G-invariant subgroup of order p^p and exponent p (otherwise, if $U/F_{i_0} \leq H/F_{i_0}$ is such a subgroup, then the p-admissible chain $\{\bar{1}\} = \bar{F}_0 < \bar{F}_1 < \cdots < \bar{F}_{i_0} < \bar{U} < \cdots < \bar{H}$ dominates strongly over $\overline{\mathcal{C}}'$, contrary to Lemmas 4 and 3(a)). Therefore, by Lemma 7, H/F_{i_0} is either absolutely regular or irregular of maximal class. Assume that H/F_{i_0} is irregular of maximal class. Then $|H/F_{i_0}| > p^{p+1}$ (otherwise, H/F_{i_0} has a G-invariant subgroup M/F_{i_0} of order p^p and exponent p, contrary to what has just been said). In that case, H/F_{i_0} has a subgroup K/F_{i_0} of maximal class and order p^{p+1} [B5]. By hypothesis, K/F_{i_0} has a characteristic subgroup L/F_{i_0} of order p^p and exponent p. Then $N/F_{i_0} = \mathrm{N}_{H/F_{i_0}}(L/F_{i_0})$ is of maximal class [B5] and order $> p^{p+1}$; then $L/F_{i_0} \leq \Phi(H/F_{i_0})$ so L/F_{i_0} is absolutely regular, a contradiction. Thus, H/F_{i_0} is absolutely regular so H/L_{i_0} is regular, by the paragraph following Remark 1. We have:

(β) H/F_{i_0} is absolutely regular, H/L_{i_0} is regular and $|\Omega_1(H/L_{i_0})| \leq p \cdot |\Omega_1(H/F_{i_0})| \leq p \cdot p^{p-1} = p^p$, and so

(γ) If $j \geq i_0 + 1$, then $|F_j| = p|\bar{F}_j| \leq p^{pj}$.

Since $\overline{\mathcal{C}}'$ is an $\mathcal{H}_p$-chain, $\Omega_{i_0+1}(\bar{H}) = \bar{F}_{i_0+1}$, so, by [B3, Remark 3],

(δ) $\Omega_{i_0+1}(H) \leq F_{i_0+1}$ hence $\Omega_{i_0+1}(H) = \Omega_{i_0+1}(F_{i_0+1})$ so, if $F_{i_0+1} < H$, then $\exp(\bar{F}_{i_0+1}) \geq p^{i_0+1}$.

Since $\exp(\bar{F}_i) \leq p^i$, we get $\exp(F_i) \leq p \cdot \exp(\bar{F}_i) \leq p^{i+1}$ for all i so

(ϵ) $\exp(F_{i_0+1}) \leq p^{i_0+2}$.

Suppose that $i_0 = 0$. In that case, as in the paragraph following (α), $\bar{H}$ is absolutely regular. Then $\bar{F}_1 = \Omega_1(\bar{H})$ is of order $< p^p$, by (δ), so $F_1 = \Omega_1(H)$ must be of order p^p and exponent p since, by the assumption, H has a G-invariant subgroup of order p^p and exponent p, and $H/\Omega_1(H) = H/F_1 \cong \bar{H}/\bar{F}_1$ is absolutely regular, by (β). In this case, by Lemma 2(b), there exists an $\mathcal{H}_p$-chain in H.

Next we let $i_0 > 0$; then $|F_1| = p^{p+1}$ and $\exp(F_1) \leq p^2$.

(ii) Let $\exp(F_{i_0+1}) < p^{i_0+1}$. Then, by ($\delta$), $F_{i_0+1} = H$ so $i_0 + 1 = n$. We also have $|H| = |F_{i_0+1}| \leq p^{p(i_0+1)}$. Let $\{1\} = L_0 < L_1 < \cdots < L_{i_0} < F_{i_0}$ be an

$\mathcal{H}_p$-chain in F_{i_0} with $|F_{i_0} : L_{i_0}| = p$ and all other indices of our chain equal p^p (Lemma 1).

If $\exp(H/L_{i_0}) = p$, then $\{1\} = L_0 < L_1 < \cdots < L_{i_0} < H$ is an $\mathcal{H}_p$-chain in H since all indices of this chain but last one are equal to p^p. Since $\exp(H/L_{i_0}) \leq p^2$, one may assume that $\exp(H/L_{i_0}) = p^2$. Therefore, since $H/F_{i_0}(= F_{i_0+1}/F_{i_0})$ is of order $\leq p^{p-1}$ and exponent p, the G-invariant subgroup $U/L_{i_0} = \Omega_1(H/L_{i_0})$ is of exponent p and index p in (the regular) group H/L_{i_0}. In that case, $\{1\} = L_0 < L_1 < \cdots < L_{i_0} < U$ is an $\mathcal{H}_p$-chain in U since only the last index of this chain is $< p^p$. Since $\exp(H/F_{i_0}) = p$ and $|H/F_{i_0}| \leq p^{p-1}$, we get $\mho_1(H) < F_{i_0}$: H is not absolutely regular in view of $i_0 > 0$. It follows that $|\mho_1(H)| \leq p^{-p} \cdot |H| \leq p^{pi_0}$. Therefore, there exists a G-invariant subgroup T_{i_0} satisfying $\mho_1(H) \leq T_{i_0} < F_{i_0}$ and $|T_{i_0}| = p^{pi_0}$. Since, in addition, we have $\exp(T_{i_0}) \leq \exp(H) = \exp(F_{i_0+1}) \leq p^{i_0}$, there exists an $\mathcal{H}_p$-chain $\{1\} = T_0 < T_1 < \cdots < T_{i_0}$ in T_{i_0} of length i_0 and all indices of this chain equal p^p (Lemma 9(a)). Then $\{1\} = T_0 < T_1 < \cdots < T_{i_0} < H$ is an $\mathcal{H}_p$-chain in H since $|H/T_{i_0}| \leq p^p$ and $\exp(H/T_{i_0}) = p$.

(iii) Let $\exp(F_{i_0+1}) = p^{i_0+1}$. By ($\delta$), $\Omega_{i_0+1}(H) = F_{i_0+1}$, and H/F_{i_0+1} is absolutely regular, by (β).

Suppose that $F_{i_0+1} < H$. To prove that there is an $\mathcal{H}_p$-chain in H, it suffices to show, in view of Lemma 2(b), that F_{i_0+1} has an $\mathcal{H}_p$-chain of length $i_0 + 1$. Let $U = \Omega_{i_0}(F_{i_0+1})(= \Omega_{i_0}(H))$; then $\exp(U) = p^{i_0}$ since H is a $\mathcal{P}_2$-group and $\exp(F_{i_0+1}) = p^{i_0+1}$. Since $L_{i_0} \leq U$, we get $|U| \geq |L_{i_0}| = p^{pi_0}$. We have $\exp(F_{i_0+1}/U) = p$ since the $\mathcal{P}_2$-group F_{i_0+1}/U is generated by elements of order p. It follows that $\mho_1(F_{i_0+1}) \leq U$. Since $i_0 > 0$, we get $|F_{i_0+1}/\mho_1(F_{i_0+1})| \geq p^p$ so $|\mho_1(F_{i_0+1})| \leq p^{-p}|F_{i_0+1}| \leq p^{pi_0}$. It follows that there is a G-invariant subgroup T_{i_0} of order p^{pi_0} such that $\mho_1(F_{i_0+1}) \leq T_{i_0} \leq U$; then $\exp(T_{i_0}) \leq \exp(U) = p^{i_0}$. By Lemma 9(a), there is an $\mathcal{H}_p$-chain $\{1\} = T_0 < T_1 < \cdots < T_{i_0}$ in T_{i_0} of length i_0, and all indices of that chain equal p^p. Then $\{1\} = T_0 < T_1 < \cdots < T_{i_0} < F_{i_0+1}$ is an $\mathcal{H}_p$-chain in F_{i_0+1} of length $i_0 + 1$ since $\exp(F_{i_0+1}/T_{i_0}) = p$ and $|F_{i_0+1}/T_{i_0}| \leq p^p$.

Now we let $F_{i_0+1} = H$; then $\exp(H) = p^{i_0+1}$ and $p^{pi_0} = |L_{i_0}| < |H| \leq p^{p(i_0+1)}$. Write $U = \Omega_{i_0}(H)$; then as in the previous paragraph, $\exp(U) = p^{i_0}$, $\exp(H/U) = p$ and $|U| \geq |L_{i_0}| = p^{pi_0}$ so $|H : U| \leq p^p$. Then there is a G-invariant subgroup T_{i_0} of order p^{pi_0} such that $\mho_1(H) \leq T_{i_0} \leq U$ since $|H : \mho_1(H)| \geq p^p$: H is not absolutely regular. We have $\exp(T_{i_0}) \leq \exp(U) = p^{i_0}$ and $\exp(H/T_{i_0}) = p$. Therefore, if $\mathcal{C}' : \{1\} = T_0 < T_1 < \cdots < T_{i_0}$ is an

$\mathcal{H}_p$-chain in T_{i_0} all of whose indices equal p^p (Lemma 9(a)), then $\{1\} = T_0 < T_1 < \cdots < T_{i_0} < H$ is an $\mathcal{H}_p$-chain in H since $|H/T_{i_0}| \le p^p$.

(iv) It remains to consider, in view of (ϵ), the case $\exp(F_{i_0+1}) = p^{i_0+2}$; then $\exp(F_{i_0}) = p^{i_0+1}$ since $\exp(F_{i_0}) \le p^{i_0+1}$ and $p^{i_0+2} = \exp(F_{i_0+1}) \le p \cdot \exp(F_{i_0})$. By ($\delta$), $\Omega_{i_0+1}(H) = \Omega_{i_0+1}(F_{i_0+1})$. By ($\gamma$), $|F_{i_0+1}| \le p^{p(i_0+1)}$.

(1iv) First suppose that $F_{i_0+1} < H$. Let $\mathcal{C}' : \{1\} = L_0 < L_1 < \cdots < L_{i_0} < \cdots < F_{i_0+1}$ be an $\mathcal{H}_p$-chain in F_{i_0+1} existing by induction. As in the paragraph following (α), H/F_{i_0} is absolutely regular so H/L_{i_0} is regular and $\Omega_1(H/L_{i_0})$ is of order $\le p^p$ (see (β)). We have $L_{i_0+1} = \Omega_{i_0+1}(F_{i_0+1})(= \Omega_{i_0+1}(H))$ since $\mathcal{C}'$ is an $\mathcal{H}_p$-chain. Also, $L_{i_0+1} < F_{i_0+1}$ in view of $\exp(L_{i_0+1}) = p^{i_0+1} < p^{i_0+2} = \exp(F_{i_0+1})$. Next, $L_{i_0+1}/L_{i_0} = \Omega_1(H/L_{i_0})$ (indeed, if $D/L_{i_0} = \Omega_1(H/L_{i_0})$, then $\exp(D) \le p^{i_0+1}$ so $D \le \Omega_{i_0+1}(H) = \Omega_{i_0+1}(F_{i_0+1}) = L_{i_0+1}$), so $F_{i_0} \le L_{i_0+1}$. It follows that H/L_{i_0+1} is absolutely regular as an epimorphic image of H/F_{i_0} (see (β)). In that case, there is an $\mathcal{H}_p$-chain in H since $\{1\} = L_0 < L_1 < \cdots < L_{i_0} < L_{i_0+1}$ is an $\mathcal{H}_p$-chain in $L_{i_0+1} = \Omega_{i_0+1}(H)$ of length $i_0 + 1$ (Lemma 2(b)).

(2iv) Now let $F_{i_0+1} = H$; then $|H| \le p^{p(i_0+1)}$ and $\exp(H) = p^{i_0+2}$. Set $U = \Omega_{i_0+1}(H)$; then $\exp(U) = p^{i_0+1}$ since H is a $\mathcal{P}_2$-group, $\exp(H/U) = p$ as above, and U contains a G-invariant subgroup L_{i_0} of order p^{pi_0} and exponent p^{i_0}. By induction, there is in U an $\mathcal{H}_p$-chain, say $\mathcal{C}' : \{1\} = K_0 < K_1 < \cdots < K_{i_0} < \cdots < U$. We have $K_{i_0+1} = \Omega_{i_0+1}(U) = U$ so $|\mathcal{C}'| = i_0 + 1$. It follows that $\{1\} = K_0 < K_1 < \cdots < K_{i_0} < K_{i_0+1} = U < H$ is an $\mathcal{H}_p$-chain in H since H/U is of order $< |H/L_{i_0}| \le p^p$ and exponent p. This completes the proof of (iv) and thereby the theorem. ∎

It is worthwhile to notice that if G is an irregular $\mathcal{P}_3$-group of order p^{p+1}, then $\Omega_1(G)$ is its characteristic subgroup of order p^p and exponent p.

The proof of Theorem 10 does not work if we suppose from the start that $H = G$ (however, the theorem is true for $H = G$). Indeed, then we cannot use induction in proper normal subgroups of G.

Theorem 11: *Let $H > \{1\}$ be a normal $\mathcal{P}$-subgroup of a p-group G and let k be fixed. Then there exists in H a chain $\mathcal{C} : \{1\} = L_0 < L_1 < \cdots < L_n = H$ of G-invariant subgroups with the following properties ($i = 1, \ldots, n$):*

(a) L_i/L_{i-1} *is of order* $\leq p^k$ *and exponent* p*, and*
(b) *either the order of* L_i *is exactly* p^{ik}*, or else* $L_i = \Omega_i(H)$*.*

In other words, there is an $\mathcal{H}_k$-chain in G. As we have noticed, the theorem is known for $k \in \{p-1, p\}$.

Proof. We proceed by induction on $|H|$. Set $\exp(H) = p^e$ and assume that $e > 1$ and $k > 1$.

(i) Suppose that H has no G-invariant subgroup of order p^k and exponent p. Then $|\Omega_1(H)| < p^k$ since H is a $\mathcal{P}_2$-group. Since H is pyramidal (Remark 1) and $\mathcal{P}_2$-group, $\{1\} < \Omega_1(H) < \cdots < \Omega_e(H) = H$ is the unique $\mathcal{H}_k$-chain in H. In what follows we assume that H has a G-invariant subgroup of order p^k and exponent p so, since H is a $\mathcal{P}_3$-group, we get $|H : \mho_1(H)| = |\Omega_1(H)| \geq p^k$.

(ii) If H is of order p^{tk} with $t \geq e$, then the theorem is true, by Lemma 9(b).

(iii) Suppose that H is of order p^{tk+s} with $t \geq e$ and $1 \leq s < k$; then $|\mho_1(H)| \leq p^{-k} \cdot |H| < p^{tk}$ since H is a $\mathcal{P}_3$-group, by hypothesis. Therefore, there is a G-invariant subgroup $U < H$ of order p^{tk} such that $\mho_1(H) < U$. We have $\exp(U) \leq p^e \leq p^t$ so, by Lemma 9(b), there is an $\mathcal{H}_k$-chain $\{1\} = U_0 < U_1 < \cdots < U_t = U$ of length t; since all indices of that chain are equal to p^k and H/U is of order $< p^k$ and exponent p, it follows that $\{1\} = U_0 < U_1 < \cdots < U_t < H$ is an $\mathcal{H}_k$-chain in H.

(iv) Suppose that $|\Omega_t(H)| = p^{tk}$ for some $t \leq e$. If $t = e$, then there is an $\mathcal{H}_k$-chain in H (Lemma 9(b) and Remark 1). Now let $t < e$; then $\exp(\Omega_t(H)) = p^t$. By Lemma 9(b), there is an $\mathcal{H}_k$-chain $\{1\} = L_0 < L_1 < \cdots < L_t = \Omega_t(H)$ of length t in $\Omega_t(H)$. Set $\bar{G} = G/L_t$. By induction, there is an $\mathcal{H}_k$-chain $\{\bar{1}\} < \bar{L}_{t+1} < \cdots < \bar{L}_{t+m} = \bar{H}$ in $\bar{H}$. Then, by Lemma 2(a), $\{1\} = L_0 < L_1 < \cdots < L_t < L_{t+1} < \cdots < L_{t+m} = H$ is an $\mathcal{H}_k$-chain in H.

(v) Suppose that $|\Omega_t(H)| > p^{tk}$ for all $t \leq e$. In particular, $|H| > p^{ek}$. Then $|H| = p^{t_0 k+s}$ for some integers t_0 and $s < k$. It follows that $p^{ek} < |H| = p^{t_0 k+s}$ so $t_0 \geq e$. As we have noticed, $|H : \mho_1(H)| \geq p^k$. Let $U_{t_0}/\mho_1(H)$ be a G-invariant subgroup of index p^s in $H/\mho_1(H)$; then $|U_{t_0}| = p^{t_0 k}$ and $\exp(U_{t_0}) \leq p^e \leq p^{t_0}$. By Lemma 9(b), there is an $\mathcal{H}_k$-chain $\{1\} = U_0 < U_1 < \cdots < U_{t_0}$ in U_{t_0} with all indices equal p^k; then $\{1\} = U_0 < U_1 < \cdots < U_{t_0} < H$ is an $\mathcal{H}_k$-chain in H.

(vi) Suppose that $|\Omega_t(H)| < p^{tk}$ for some positive $t \leq e$. Let t be minimal subject to that inequality. Then $t > 1$ since H has a G-invariant subgroup of order p^k and exponent p. By the choice of t, we get $|\Omega_{t-1}(H)| \geq p^{(t-1)k}$.

In view of (iv), one may assume that $|\Omega_{t-1}(H)| > p^{(t-1)k}$. It follows that $|\Omega_t(H)/\Omega_{t-1}(H)| < p^{k-1}$ so that $|\Omega_1(H/\Omega_{t-1}(H))| < p^{k-1}$ (indeed, if $A/\Omega_{t-1}(H) \leq H/\Omega_{t-1}(H)$ is of order p^{k-1} and exponent p, then $A \leq \Omega_t(H)$ and A is of order $\geq p^{tk} > |\Omega_t(H)|$, which is not the case). Thus, the pyramidal quotient group $H/\Omega_{t-1}(H)$ (Remark 1) has no normal subgroup of order p^k and exponent p. So, setting $\bar{G} = G/\Omega_t(H)$, we conclude that $\{\bar{1}\} < \Omega_1(\bar{H}) < \cdots < \bar{H}$ is an $\mathcal{H}_k$-chain in $\bar{H}$. Therefore, in view of Lemma 2(a), it suffices to prove that there is in $\Omega_t(H)$ an $\mathcal{H}_k$-chain of length t. Assume that this is false; then the length of our chain is $> t$ so let $\{1\} = T_0 < T_1 < \cdots < T_t < \cdots < \Omega_t(H)$ be an $\mathcal{H}_k$-chain in $\Omega_t(H)$. Since $|T_t| < |\Omega_t(H)| < p^{tk}$, it follows that $T_t = \Omega_t(\Omega_t(H)) = \Omega_t(H)$, which is a contradiction.

Since all possibilities for $\Omega_t(H)$ are considered, the proof is complete. ∎

It is possible to prove Theorem 10 in the same way as Theorem 11 but the presented proof is shorter. However, the argument in the proof of Theorem 10 is more universal since it also proves [B3, Theorem 1], which is not the case for argument in the proof of Theorem 11.

Let G be an arbitrary p-group of order p^n. Then $W = G \times E$, where E is the elementary abelian p-group of order $p^{n(k-1)}$, has a chain of normal subgroups of length n all of whose factors are of order p^k and exponent p.

Let G be an abelian p-group of exponent $p^e > p$. We claim that G is homocyclic if and only if $\mho_{e-1}(G) = \Omega_1(G)$. Suppose that the last equality holds. Set $|\Omega_1(G)| = p^d$. Then $G = Z_1 \times \cdots \times Z_d$, where Z_i are all cyclic. In that case, $\mho_{e-1}(G) = \mho_{e-1}(Z_1) \times \cdots \times \mho_{e-1}(Z_d)$ is of order p^d so $|\mho_{e-1}(Z_i)| = p$. It follows that $|Z_i| = p^e$ for all i, hence G is homocyclic. The converse assertion is obvious.

Remark 4: Given a normal subgroup H in G, let $Ch_k(H)$ be the number of $\mathcal{H}_k$-chains in H. We claim that if $\mathcal{C} : \{1\} = L_0 < L_1 < \cdots < L_n = G$ is an $\mathcal{H}_k$-chain in G, then $Ch_k(G) \geq Ch_k(L_j)$ for all $j \leq n$. Assume that $\{1\} = M_0 < M_1 < \cdots < M_j = L_j$ is an $\mathcal{H}_k$-chain in L_j. Then $\mathcal{C}' : \{1\} = M_0 < M_1 < \cdots < M_j = L_j < L_{j+1} < \cdots < L_n = G$ is an $\mathcal{H}_k$-chain. This is true for $j \leq i_0$. Now let $j > i_0$. Let $i_0 < i_2 \leq j$. Then $M_{i_2} = \Omega_{i_2}(M_j) = \Omega_{i_2}(L_j) = \Omega_{i_2}(\Omega_j(G)) = \Omega_{i_2}(G)$ so $\mathcal{C}'$ is an $\mathcal{H}_k$-chain in G, and we are done.

Proposition 12: *An abelian p-group G has exactly one $\mathcal{H}_k$-chain if and only if $|\Omega_1(G)| \leq p^k$.*

Proof. If $|\Omega_1(G)| \le p^k$, then $\{1\} < \Omega_1(G) < \cdots < \Omega_e(G) = G$ ($\exp(G) = p^e$) is the unique $\mathcal{H}_k$-chain in the pyramidal group G.

It remains to prove that if G has exactly one $\mathcal{H}_k$-chain, then $|\Omega_1(G)| \le p^k$. Assume that G is a counterexample of minimal order, then $k > 1$, $e > 1$ and $|\Omega_1(G)| > p^k$. By Theorem 11, there is in G an $\mathcal{H}_k$-chain $\mathcal{C} : \{1\} < L_1 < \cdots < L_n = G$, and this chain is unique, by hypothesis. By assumption, $|L_1| = p^k$. In view of Remark 4, each member L_j of the chain $\mathcal{C}$ has exactly one $\mathcal{H}_k$-chain so, by induction, $|\Omega_1(L_j)| = p^k$ for $1 \le j < n$. Set $i_0 = i_0(\mathcal{C})$.

Suppose that $n > i_0 + 1$. Then $\Omega_{i_0+1}(G) = L_{i_0+1} < G$ so $\Omega_1(G) < L_{i_0+1}$. By induction, $|\Omega_1(L_{i_0+1})| = p^k$, a contradiction since $\Omega_1(G) = \Omega_1(L_{i_0+1})$. Thus, $n \le i_0 + 1$.

Write $\bar{G} = G/L_1$. If $\overline{\mathcal{C}}_1$ is an $\mathcal{H}_k$-chain in $\bar{G}$, then its inverse image $\mathcal{C}_1$ in G is also an $\mathcal{H}_k$-chain in G. Indeed, $i_0(\overline{\mathcal{C}}_1) = i_0(\overline{\mathcal{C}}) = i_0 - 1$ so all indices of the chain $\mathcal{C}_1$ apart for possibly the last one, are equal to p^k, and our claim follows. We conclude that $\overline{\mathcal{C}} : \{\bar{1}\} < \bar{L}_2 < \cdots < \bar{L}_n = \bar{G}$ is the unique $\mathcal{H}_k$-chain in $\bar{G}$ (Lemma 4) so $|\Omega_1(\bar{G})| \le p^k$. It follows that $\Omega_1(\bar{G}) = \bar{L}_2$ so $\Omega_1(G) \le L_2$; then $\Omega_1(G) = \Omega_1(L_2)$.

Assume that $L_2 < G$. Since L_2 has only one $\mathcal{H}_k$-chain (Remark 4), we get, by induction, $|\Omega_1(L_2)| = p^k$ so $|\Omega_1(G)| = p^k$, contrary to the assumption.

Now let $L_2 = G$. Then $|G| = |L_1||L_2/L_1| \le p^{2k}$, $p^k < |\Omega_1(G)| = |G/\mho_1(G)|$ and $\mho_1(G) \le \Omega_1(G)$. It follows that $|\mho_1(G)| < p^k$. Let $\mho_1(G) < M_1 < \Omega_1(G)$, where $|M_1| = p^k$. Since there are > 1 possibilities to choose M_1, one may assume from the start that $M_1 \ne L_1$. Then $\mathcal{C}' : \{1\} < M_1 < G$ is an $\mathcal{H}_k$-chain in G and $\mathcal{C}' \ne \mathcal{C}$, a final contradiction. ∎

Let $p > 3$ and let P be a Sylow p-subgroup of the symmetric group of degree p^2. Set $G = P/\mho_1(P)$. Let H be the unique abelian subgroup of index p in G and $k > 1$ a proper divisor of $p - 1$. Then there is only one $\mathcal{H}_k$-chain in H and $\Omega_1(H) = H$ is of order $|\Omega_1(H)| = |H| = p^{p-1} > p^k$.

Supplement 1 to Proposition 12: *Let $H \lhd G$ be abelian such that all subgroups of H are normal in G. There is exactly one $\mathcal{H}_k$-chain in H if and only if $|\Omega_1(H)| \le p^k$.*

Supplement 2 to Proposition 12: *If G is an abelian p-group with $|\Omega_1(G)| > p^k$, then $Ch_k(G) \ge p + 1$ (see Remark 4).*

Proof. Suppose that G is a counterexample of minimal order. Let $\mathcal{C}$ be an $\mathcal{H}_k$-chain in G. Then, by Remark 4, we must have $Ch_k(L_j) = 1$ for all $j < n$.

Assume that $n > i_0 + 1$. Then $L_{i_0+1} = \Omega_{i_0+1}(G) < G$ so $\Omega_1(G) = \Omega_1(L_{i_0+1})$. Since $Ch_k(L_{i_0+1}) = 1$, we get $|\Omega_1(L_{i_0+1})| = p^k$ (Proposition 12), which is not the case.

Thus, $n \le i_0 + 1$. It follows from the proof of Proposition 12 that $Ch_k(G) \ge Ch_k(G/L_1)$ so $Ch_k(G/L_1) = 1$, by induction. By Proposition 12, $|\Omega_1(G/L_1)| \le p^k$ so $\Omega_1(G/L_1) = L_2/L_1$, and we conclude that $\Omega_1(G) = \Omega_1(L_2)$. If $L_2 < G$, then $Ch_k(L_2) = 1$ so $|\Omega_1(L_2)| = p^k$ (Proposition 12), contrary to the assumption.

We conclude that $L_2 = G$. Then $|G| \le p^{2k}$ and, as in the proof of Proposition 12, $|\mho_1(G)| < p^k$. If $\mho_1(G) < M_1 < \Omega_1(G)$ is such that $|M_1| = p^k$, then $\{1\} \le M_1 < G$ is an $\mathcal{H}_k$-chain in G. Since the number of possibilities for the choice of M_1 is $\ge p + 1$, we get $Ch_k(G) \ge p + 1$ (Remark 4), and G is not a counterexample. ∎

It is interesting to classify the abelian p-groups G with $Ch_k(G) = p + 1$. It follows from the proof of the previous supplement that then $|\mho_1(G)| = p^{k-1}$ and $|\Omega_1(G)| = p^{k+1}$.

Remark 5: Let us show that, for a nonabelian p-group G, $Ch_1(G) = p + 1$ if and only if G is of maximal class. Clearly, if G is of maximal class, then $Ch_1(G) = p + 1$. Now, supposing that $Ch_1(G) = p + 1$, we prove by induction on G that G is of maximal class. Set $|G| = p^m$. The assertion holds for $m = 3$; so assume that $m > 3$. Every $\mathcal{H}_1$-chain in G is not more than a principal series of G. It follows that $|G/G'| = p^2$ so $\mathrm{Z}(G) \le G'$. Each normal subgroup of G is a member of some $\mathcal{H}_1$-chain. We conclude that G' has only one G-invariant subgroup of order p which we denote by R_1. Since $m > 3$ and $|G/G'| = p^2$, G/R_1 is nonabelian. Obviously, $Ch_1(G/R_1) = p + 1$ so, by induction, G/R_1 is of maximal class, and we have $|\mathrm{Z}(G/R_1)| = p$. It remains to show that $|\mathrm{Z}(G)| = p$. Assume that this is false. Then $|\mathrm{Z}(G)| = p^2$ and $\mathrm{Z}(G)$ is cyclic (otherwise, $Ch_1(G) \ge (p+1)^2 > p + 1$). Since G is not of maximal class, it has a normal abelian subgroup, say R, of type (p, p). Since $R_1 < R$, we get $R/R_1, R_2/R_1 \le \mathrm{Z}(G/R_1)$ so $|\mathrm{Z}(G/G_1)| \ge p^2$, a final contradiction.

In conclusion we consider an arbitrary p-group G without $\mathcal{H}_p$-chains. By Theorem 11, G must be irregular. Let $\mathcal{M}$ be the set of all normal subgroups

H of G such that there is an $\mathcal{H}_p$-chain, say $\mathcal{C}_H$, in H (as a normal subgroup of G). Let $\mathcal{M}_0$ be the set of all $H \in \mathcal{M}$ such that, whenever $H_1 \in \mathcal{M}$ with an $\mathcal{H}_p$-chain, say $\mathcal{C}_{H_1}$, then, with respect to lexicographic ordering, the sequence $|L_1|, |L_2 : L_1|, \dots, |L_n : L_{n-1}|$ of indices of the chain $\mathcal{C}_H$ is greater or equal than the sequence $|M_1|, |M_1 : M_0|, \dots, |M_s : M_{s-1}|$ of indices of the chain $\mathcal{C}_{H_1}$ (so that we compare only $\mathcal{H}_p$-chains). Thus, if $H, H_1 \in \mathcal{M}_0$, then $i_0(\mathcal{C}_H) = i_0(\mathcal{C}_{H_1})$, $|\mathcal{C}_H| = |\mathcal{C}_{H_1}|$, $|H| = |H_1|$ and corresponding indices of these chains are equal. In what follows we use the notation introduced in this paragraph.

Let G be a p-group of maximal class and order $> p^3$ and let $R \lhd G$ with $|G : R| = p^4$. Then G/R has the unique abelian subgroup G_1/R of index p. This G_1 is called the **fundamental** subgroup of G. Clearly, G_1 is characteristic in G.[2] If, in addition, $|G| > p^{p+1}$, then G_1 is the unique regular maximal subgroup of G; all other maximal subgroups of G are irregular of maximal class.

Proposition 13: *Let a p-group G have no $\mathcal{H}_p$-chains and let $H \in \mathcal{M}_0$, $p > 2$. Suppose that H has no normal subgroup of order p^p and exponent p, or, what is the same, $i_0(\mathcal{C}_H) = 0$. Then G is of maximal class and order $\geq p^{2p}$ and H is either absolutely regular or of maximal class.*

(a) *If H is absolutely regular, then $H = G_1$, the fundamental subgroup of G, and $\mathcal{M}_0 = \{H\}$.*

(b) *Suppose that H is irregular of maximal class. Then $|G : H| = p$, $|H| = p^{2p-1}$ and $|\Omega_1(H)| = p^{p-1}$. In that case, $G_1 \notin \mathcal{M}_0$.*

Proof. By Theorem 11, G is irregular. We also have $|G| > p^{p+1}$ (otherwise, G has an $\mathcal{H}_p$-chain as Remark 3 shows). It follows from Lemma 7 that H is either absolutely regular or irregular of maximal class. If there is $R \lhd G$ of order p^p and exponent p, then the $\mathcal{H}_p$-chain $\mathcal{C}_R : \{1\} < R$ dominates over $\mathcal{C}_H$, contrary to the choice of H. Therefore, by Lemma 7 and Theorem 11, G is of maximal class. Since G has a normal subgroup of order p^{p-1} and exponent p, we get $|\Omega_1(H)| = p^{p-1}$. It is worthwhile to notice that any normal subgroup K of index $> p$ in G is contained in $\Phi(G)$ so absolutely regular; then also $K < G_1$. Since G has no $\mathcal{H}_p$-chain, we get $|G| > p^{2p-1}$ (Remark 3).

[2] Using so defined subgroup G_1, it is easy to construct an $\mathcal{H}_k$-chain in G for each k, $1 < k < p$ (see Remark 3). Indeed, let $|G_1| = p^n$, where $n = kt + s$, $0 \leq s < k$. Let L_i be a G-invariant subgroup of order p^{ki} in G_1, $i \leq t$. Then $\mathcal{C} : \{1\} = L_0 < L_1 < \cdots < L_t < G$ is an $\mathcal{H}_k$-chain in G. So constructed chain $\mathcal{C}$ is a unique $\mathcal{H}_k$-chain in G if and only if $s > 0$ (for $s = 0$, the number of $\mathcal{H}_k$-chains in G equals $p + 1$).

Let H be absolutely regular. Then $|G : H| = p$ (otherwise, $H < G_1$ and there is an $\mathcal{H}_p$-chain in G_1). Since G_1 is the unique regular maximal subgroup of G, we get $H = G_1$. Suppose, in addition, that $H_0 \in \mathcal{M}_0 - \{H\}$. In that case, H_0 is irregular of maximal class and index p in G. Since H_0 has an $\mathcal{H}_p$-chain, we get $|H| \le p^{2p-1}$ so we have $|H_0| = p^{2p-1}$ since $|G : H_0| = p$ and $G| > p^{2p-1}$. In that case, the last index of the $\mathcal{H}_p$-chain of H_0 equals p^p so it is not equal to every index of the $\mathcal{H}_p$-chain of the absolutely regular group H, and this is a contradiction. Thus, we have $\mathcal{M}_0 = \{1\}$, completing the proof of (a).

Now let H be irregular of maximal class. Then, as we have noticed already, $|G : H| = p$. By Remark 3, since $|G| \ge p^{2p}$, we get $|H| = p^{2p-1}$. Since G has no normal subgroup of order p^p and exponent p^p, we get $|\Omega_1(H)| = p^{p-1}$. Since the $\mathcal{H}_p$-chain of G_1 has no index $= p^p$, we get $G_1 \notin \mathcal{M}_0$, completing the proof of (b) and thereby the proposition. ∎

Proposition 14: *Suppose that a p-group G has no $\mathcal{H}_p$-chains. Let $H \in \mathcal{M}_0$ and let*

$$\mathcal{C} = \mathcal{C}_H : \{1\} = L_0 < L_1 < \cdots < L_n = H$$

be an $\mathcal{H}_p$-chain in H with $i_0 = i_0(\mathcal{C}) > 0$. Write $U = L_{i_0}$ and $\bar{G} = G/U$. Then $|\mathcal{C}| > i_0$ and one of the following holds:

(a) *$\bar{G}$ is absolutely regular. Then $\exp(\bar{G}) > p$. In that case, $L_{i_0+1} < \Omega_{i_0+1}(G)$ and, if $|\mathcal{C}_H| > i_0 + 1$, then $\exp(L_{i_0+1}) = p^{i_0+1}$, $L_{i_0+1} < \Omega_{i_0+1}(G)$ and $L_{i_0+2}/L_{i_0+1} < \Omega_1(G/L_{i_0+1})$.*

(b) *$\bar{G}$ is irregular of maximal class. Then $\bar{H}$ is either absolutely regular or irregular of maximal class.* (b1) If $|\bar{G}| = p^{p+1}$, then all maximal subgroups of $\bar{G}$ are absolutely regular so $|\Omega_1(\bar{G})| = p^{p-1}$. In that case, $L_{i_0} < \Omega_{i_0}(G)$.

(b2) Let $\bar{H}$ be irregular of maximal class. Then $|G : H| = p$, $|\Omega_1(\bar{H}) = p^{p-1}$, $|\bar{G}| \le p^{2p}$ and $|\mathcal{C}_H| = i_0 + 2$.

Proof. Write $i_0 = i_0(\mathcal{C})$ and $U = L_{i_0}$. If F/U is normal subgroup of order p^p and exponent p in G/U, then the $\mathcal{H}_p$-chain in F strongly dominates over $\mathcal{C}$, which is a contradiction. Therefore, $\bar{G} = G/U$ has no normal subgroup of order p^p and exponent p so it is either absolutely regular or irregular of maximal class (Lemma 7). Assume that $|C| = i_0$. Take in $\bar{G}$ a normal subgroup $\bar{F}$ of order p. Then $\{1\} = L_1 < L_1 < \cdots < L_{i_0} = H < F$ is an $\mathcal{H}_p$-chain in F and it strongly dominates over $\mathcal{C}$, a contradiction. Thus, $|\mathcal{C}| > i_0$.

(a) Suppose that G/U is absolutely regular; then $\exp(G/U) > p$ (otherwise, G has an $\mathcal{H}_p$-chain $\{1\} = L_1 < L_1 < \cdots < L_{i_0} = U < G$). Since L_{i_0+1} has an $\mathcal{H}_p$-chain of length $i_0 + 1$ and G/L_{i_0+1} is absolutely regular, it follows that $L_{i_0+1} < \Omega_{i_0+1}(G)$ (otherwise, by Lemma 2(b), G has an $\mathcal{H}_p$-chain). Next, suppose that $|\mathcal{C}_H| > i_0 + 1$. Consider the subgroup $W = L_{i_0+2}$. We have $\Omega_{i_0+1}(H) = L_{i_0+1}(< H)$ so $\exp(L_{i_0+1}) = p^{i_0+1}$. Assume that $W/L_{i_0+1} = \Omega_1(G/L_{i_0+1})$. Let $x \in G - L_{i_0+1}$ be of minimal order and let $o(xL_{i_0+1}(G) = p$ in G/L_{i_0+1}, then, by what has been said already, $o(x) \leq p^{i_0+1}$ and $x^p \in L_{i_0+1}$. In that case, by construction, $x \in W = L_{i_0+2}$. However, $x \in \Omega_{i_0+1}(W) = L_{i_0+1}$, contrary to the choice of x. Thus, $W/L_{i_0+1} < \Omega_1(G/L_{i_0+1})$, completing this case.

(b) Suppose that $\bar{G} = G/U$ is irregular of maximal class. Then $\bar{H}$ is either absolutely regular or of maximal class.

Let $|\bar{G}| = p^{p+1}$. If $\bar{H}_1 < \bar{G}$ is of order p^p and exponent p, then $\{1\} = L_0 < L_1 < \cdots < L_{i_0} < H_1 < G$ is a $\mathcal{H}_p$-chain in G (all indices of that chain, apart of the last one, equal p^p), so, comparing indices of that chain with indices of the chain $\mathcal{C}$, we get $H \notin \mathcal{M}_0$, a contradiction. Thus, all maximal subgroups of $\bar{G}$ are absolutely regular so $\Omega_1(\bar{G}) = \Phi(\bar{G})$ is of order p^{p-1} and exponent p. Since G has no $\mathcal{H}_p$-chain, it follows that $L_{i_0} < \Omega_{i_0}(G)$ (otherwise, G has an $\mathcal{H}_p$-chain, whose (i_0+1)th member coincides with the inverse image of $\Omega_1(\bar{G})$ in G, and the following member is G), completing the proof of (b1).

Now let $\bar{H}$ be irregular of maximal class; then $|G : H| = |\bar{G} : \bar{H}| = p$. Since $\bar{H}$ has no G-invariant subgroup of order p^p and exponent p, we get $|\Omega_1(\bar{H})| = p^{p-1}$ since $\bar{H}$ has an $\mathcal{H}_p$-chain. All remaining assertions follow from Proposition 13. ∎

Metacyclic p-groups, $p > 2$, regular so they have an $\mathcal{H}_2$-chain (Theorem 11). This is not the case for metacyclic 2-groups.

Proposition 15: *The following conditions for a metacyclic 2-group G of order $\geq 2^4$ are equivalent:*

(a) *G has no $\mathcal{H}_2$-chain.*

(b) *There is $k \geq 0$ such that $|\Omega_i(G)| = 2^{2i}$ for all $i \leq k$ and $G/\Omega_k(G)$ is of maximal class and order $\geq 2^4$.*

Proof. Let the set $\mathcal{M}_0$ be such as in Propositions 13 and 14. Take $H \in \mathcal{M}_0$ and let $\mathcal{C} = \mathcal{C}_H$ be a $\mathcal{H}_2$-chain in H. Set $i_0 = i_0(\mathcal{C})$. Then $U = L_{i_0} = \Omega_{i_0}(G)$

so G/U is of maximal class, by the above and Lemma 2(b). If $G/U \cong \mathrm{Q}_8$, then $\{1\} < \Omega_1(G) < \cdots < U < \Omega_{i_0+1}(G) < G$ is an $\mathcal{H}_2$-chain in G, a contradiction. Now assume that $G/U \cong \mathrm{D}_8$. Let $H/U < G/U$ be abelian of type $(2, 2)$. Then $\{1\} < \Omega_1(G) < \cdots < U < H < G$ is an $\mathcal{H}_2$-chain in G, a contradiction. Thus, $|G/U| \geq 2^4$. Clearly, $|\Omega_i(G)| = 2^{2i}$ for all $i \leq i_0$.

Let $|G/U| > 2^3$. It remains to show that G has no $\mathcal{H}_2$-chain. Assume that $\mathcal{C}$ is an $\mathcal{H}_2$-chain as in Definition 1. Since G/U is of maximal class, we get $\Omega_{i_0+2}(G) = G$ so $L_{i_0+2} = G$. Since $|L_{i_0+2}| \leq 2^{2k+3} < 2^{2k+4} \leq G$, we get a contradiction. ∎

Supplement to Proposition 15: *Let G be a metacyclic p-group such that $Ch_2(G) > 1$ (see Remark 4). Then $p = 2$ and there is k such that $|\Omega_i(G)| = 2^{2i}$ and $G/\Omega_{2i}(G)$ is dihedral of order 8 for all $i \leq k$ (in the last case, $Ch_2(G) = 2$).*

This follows easily from the proof of Proposition 15.

PROBLEMS. Below, $H > \{1\}$ is a normal subgroup of a p-group G.

1. Study the structure of G provided there exists only one $\mathcal{H}_2$-chain in G.
2. Let G be a group of order p^{pk} such that $\Omega_k(G) = G$. Describe the structure of G provided $\exp(G) > p^k$.
3. Find an algorithm producing all $\mathcal{H}_k$-chains in abelian p-groups.
4. Classify the 2-groups which have no $\mathcal{H}_2$-chains.
5. Given a natural number k, a chain $\mathcal{C}_0 : H = H_0 > H_1 > \cdots > H_n = \{1\}$ of G-invariant subgroups is said to be a lower k-admissible chain in H provided H_{i-1}/H_i is of order $\leq p^k$ and exponent p ($i = 1, \ldots, n$). The above chain is said to be a lower $\mathcal{H}_k$-chain in G if, whenever $|H/H_i| < p^{ki}$, then $H_i = \mho_i(H)$. (i) Is it true that, whenever H is a lower pyramidal (see Remark 1), it possesses a lower $\mathcal{H}_k$-chain? (ii) Study the p-groups without lower $\mathcal{H}_p$-chains.
6. Suppose that H is a $\mathcal{P}_2$-subgroup such that all sections of H are pyramidal. Is it true that there exists in H an $\mathcal{H}_k$-chain for any k?
7. Does there exist in H an $\mathcal{H}_p$-chain if all sections of H of order p^{p+1} are $\mathcal{P}$-groups?
8. Suppose that p-groups G and G_0 are lattice isomorphic and G has an $\mathcal{H}_p$-chain. Is it true that also G_0 has an $\mathcal{H}_p$-chain?

9. Suppose that a p-group H has an $\mathcal{H}_k$-chain, say $\mathcal{C}$. Now let $H \triangleleft G$, where G is a p-group. Find sufficient conditions for existing an $\mathcal{H}_k$-chain in H (as a normal subgroup in G).
10. Is it true that the number of $\mathcal{H}_k$-chains in any abelian p-group is congruent with 1 $\pmod p$?

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References

[B2] Y. Berkovich, *On subgroups and epimorphic images of finite p-groups,* Journal of Algebra **248** (2002), 472–553.

[B3] Y. Berkovich, *Alternate proofs of two theorems of Philip Hall on finite p-groups, and some related results,* Journal of Algebra **294** (2005), 463–477.

[B4] Y. Berkovich, *A property of p-groups of order* $p^{p(e+1)}$ *and exponent* p^e, Glasnik Matematicki **40**(60) (2005), 51–58.

[B1] Y. Berkovich, *Groups of Prime Power Order,* Volume I, Walter de Gruyter, Berlin, 2008.

[B5] N. Blackburn, *On a special class of p-groups,* Acta Mathematica **100** (1958), 45–92.

[H] P. Hall, *On a theorem of Frobenius,* Proceedings of the London Mathematical Society (2) **40** (1936), 468–501.

[Hup] B. Huppert, *Endliche Gruppen,* Bd. I, Springer, Berlin, 1967.

[M] A. Mann, *The power structure of p-groups,* Journal of Algebra **42** (1976), 121–135.

[LM] A. Lubotzky and A. Mann, *Powerful p-groups 1,* Journal of Algebra **105** (1987), 484–505.